

Observables, evolution equation, and stationary states equation in the joint probability representation of quantum mechanics

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Abstract

Symplectic and optical joint probability representations of quantum mechanics are considered, in which the functions describing the states are the probability distributions with all random arguments (except the argument of time). The general formalism of quantizers and dequantizers determining the star product quantization scheme in these representations is given. Taking the Gaussian functions as the distributions of the tomographic parameters the correspondence rules for most interesting physical operators are found and the expressions of the dual symbols of operators in the form of singular and regular generalized functions are derived. Evolution equations and stationary states equations for symplectic and optical joint probability distributions are obtained.

Keywords: Quantum tomography, optical tomogram, symplectic tomogram, joint probability distribution, correspondence rules for operators, symbols of operators.

1 Introduction

In Ref. [1] the probability representation of quantum states was suggested (for a review see [2]). According to this representation the states of quantum systems are associated with fair probability distributions called quantum tomograms. The density operators of the quantum states can be determined from the tomograms, and consequently, the tomograms contain the complete information of the quantum properties equivalent to the information embraced in all of the forms of the density operators like Wigner function [3], Husimi function [4], Glauber-Sudarshan function [5, 6].

Initially the quantum optical tomogram $w(X, \theta)$ was introduced as a tool for measuring the quantum state of radiation [7, 8]. Generalizing the optical tomography technique the symplectic tomography was introduced [9], and the evolution equation for symplectic tomograms was found in [1, 10]. The problem of dual symbols of physical observables in the symplectic tomography representation was considered in Ref. [11]. Evolution equations for optical tomograms of spinless quantum systems were obtained in Refs. [12, 13]. For the particles with spin it was done in [14, 15]. The correspondence rules and dual symbols of operators in the optical tomography representation were obtained in [16].

The quantum state tomograms depend on extra parameters, for example, the optical tomogram $w(X, \theta)$ [7, 8] depend on the random position called quadrature component and the parameter θ called local oscillator phase. It was pointed out [17, 18] that the tomogram can be interpreted as a conditional probability distribution denoted as $w(X, \theta) \equiv w(X|\theta)$, and such an interpretation provides the possibility to introduce the joint probability distribution of two random variables $\tilde{w}(X, \theta)$, which determines the optical tomogram via Bayes' formula [19].

The symplectic tomogram $M(X, \mu, \nu)$ [9] represents the distribution function of the position quadrature X of rotated and squeezed (stretched) phase plane determined by the parameters μ and ν . Thus, $M(X, \mu, \nu) \equiv M(X|\mu, \nu)$ is a conditional distribution function of the variable X under the condition of given (μ, ν) , and if the distribution function for μ and ν is known, we can introduce the joint probability distribution $\tilde{M}(X, \mu, \nu)$ of three random variables [17, 18]. Other tomographic schemes (like, e.g.,

spin tomography) also enable to construct joint probability representations with all random variables (indices).

The aim of this paper is derivation of correspondence rules and symbols of operators in the joint probability representation; and foundation of evolution equations and stationary states equations for the joint probability distributions.

The paper is organized as follows. In Section 2 we introduce the joint probability representation of states of quantum systems using the general formalism of quantizers and dequantizers. In Section 3 we discuss correspondence rules for the operators, symbols of operators, the evolution equation, and the equation of stationary states in general case for arbitrary quantizer and dequantizer. In Section 4 we consider a specific example of symplectic joint probability representation in the N -dimensional case using shifted and scaled Gaussian distribution function for the tomographic parameters $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$. We find correspondence rules of operators, symbols of operators as singular and regular generalized functions, evolution equation, and energy levels equation for joint probability distribution of states. In Section 5 the optical joint probability representation with the distribution function for the phase vector $\boldsymbol{\theta}$ in the form of the weighted sum of shifted and scaled Gaussian functions is presented. Conclusions are given in Section 6.

2 Joint probability representation of states of quantum systems

Let the density matrix $\hat{\rho}(t)$ dependent on time t and normalized by the condition $\text{Tr}\{\hat{\rho}(t)\} = 1$ corresponds to the state of the quantum system, then in the tomographic representation this state is described by the tomographic distribution function $\mathcal{F}(x, \eta, t)$ normalized by the condition

$$\int \mathcal{F}(x, \eta, t) dx = 1, \quad (1)$$

where x is a set of distribution variables and η is a set of parameters of corresponding tomography. According to the star product scheme (see [20]), the tomogram is associated with the density matrix in the following way:

$$\mathcal{F}(x, \eta, t) = \text{Tr} \left\{ \hat{\rho}(t) \hat{U}_{\mathcal{F}}(x, \eta) \right\}, \quad \hat{\rho}(t) = \int \hat{D}_{\mathcal{F}}(x, \eta) \mathcal{F}(x, \eta, t) dx d\eta, \quad (2)$$

where $\hat{U}_{\mathcal{F}}(x, \eta)$ and $\hat{D}_{\mathcal{F}}(x, \eta)$ are dequantizer and quantizer operators for appropriate tomographic scheme.

If we have spinless quantum system in the N -dimensional space, then dequantizer and quantizer for the optical tomography [16] equal

$$\hat{U}_w(\mathbf{X}, \boldsymbol{\theta}) = |\mathbf{X}, \boldsymbol{\theta}\rangle \langle \mathbf{X}, \boldsymbol{\theta}| = \prod_{\sigma=1}^N \delta \left(X_{\sigma} - \hat{q}_{\sigma} \cos \theta_{\sigma} - \hat{p}_{\sigma} \frac{\sin \theta_{\sigma}}{m_{\sigma} \omega_{\sigma}} \right), \quad (3)$$

$$\hat{D}_w(\mathbf{X}, \boldsymbol{\theta}) = \int \prod_{\sigma=1}^N \frac{\hbar |y_{\sigma}|}{2\pi m_{\sigma} \omega_{\sigma}} \exp \left\{ i y_{\sigma} \left(X_{\sigma} - \hat{q}_{\sigma} \cos \theta_{\sigma} - \hat{p}_{\sigma} \frac{\sin \theta_{\sigma}}{m_{\sigma} \omega_{\sigma}} \right) \right\} d^N y, \quad (4)$$

where m_{σ} and ω_{σ} are constants that have dimensions of mass and frequency and are chosen for reasons of convenience for the Hamiltonian of the quantum system under study, $|\mathbf{X}, \boldsymbol{\theta}\rangle$ is an eigenfunction of the operator $\hat{\mathbf{X}}(\boldsymbol{\theta})$ with components $\hat{X}_{\sigma} = \hat{q}_{\sigma} \cos \theta_{\sigma} + (\hat{p}_{\sigma} \sin \theta_{\sigma}) / (m_{\sigma} \omega_{\sigma})$ corresponding to the eigenvalue \mathbf{X} , where \hat{q}_{σ} and \hat{p}_{σ} are the canonical position and momentum operators.

For the symplectic tomography the quantizer and dequantizer [21] can be written as:

$$\hat{U}_M(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = |\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}\rangle \langle \mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}| = \prod_{\sigma=1}^N \delta(X_{\sigma} - \hat{q}_{\sigma} \mu_{\sigma} - \hat{p}_{\sigma} \nu_{\sigma}), \quad (5)$$

$$\hat{D}_M(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \prod_{\sigma=1}^N \frac{m_{\sigma}\omega_{\sigma}}{2\pi} \exp \left\{ i \sqrt{\frac{m_{\sigma}\omega_{\sigma}}{\hbar}} (X_{\sigma} - \hat{q}_{\sigma}\mu_{\sigma} - \hat{p}_{\sigma}\nu_{\sigma}) \right\}, \quad (6)$$

where $|\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}\rangle$ is an eigenfunction of the operator $\hat{\mathbf{X}}(\boldsymbol{\mu}, \boldsymbol{\nu})$ with components $\hat{X}_{\sigma} = \mu_{\sigma}\hat{q}_{\sigma} + \nu_{\sigma}\hat{p}_{\sigma}$ corresponding to the eigenvalue \mathbf{X} .

Symplectic and optical tomograms can also be found from the Wigner function $W(\mathbf{q}, \mathbf{p}, t)$:

$$M(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}, t) = \int \prod_{\sigma=1}^N \delta(X_{\sigma} - \mu_{\sigma}q_{\sigma} - \nu_{\sigma}p_{\sigma}) W(\mathbf{q}, \mathbf{p}, t) d^N q d^N p, \quad (7)$$

$$w(\mathbf{X}, \boldsymbol{\theta}, t) = \int \prod_{\sigma=1}^N \delta \left(X_{\sigma} - q_{\sigma} \cos \theta_{\sigma} - p_{\sigma} \frac{\sin \theta_{\sigma}}{m_{\sigma}\omega_{\sigma}} \right) W(\mathbf{q}, \mathbf{p}, t) d^N q d^N p, \quad (8)$$

which, in turn, is determined by the density matrix in the position representation by the well-known formula

$$W(\mathbf{q}, \mathbf{p}, t) = \frac{1}{(2\pi\hbar)^N} \int \rho(\mathbf{q} + \mathbf{u}/2, \mathbf{q} - \mathbf{u}/2, t) e^{-i\mathbf{p}\mathbf{u}/\hbar} d^N u. \quad (9)$$

Following by Refs. [17, 18] let us introduce the joint probability distribution function describing the state of a physical system. For this aim we note that if the set of parameters η is chosen randomly with a distribution function $P(\eta)$ normalized by the condition

$$\int P(\eta) d\eta = 1, \quad (10)$$

then, according to Bayes' formula [19], dependent (generally speaking) on time joint probability distribution $\tilde{\mathcal{F}}(x, \eta, t)$ of the two sets of random variables X and η will be equal

$$\tilde{\mathcal{F}}(x, \eta, t) = \mathcal{F}(x, \eta, t) P(\eta). \quad (11)$$

Due to normalization property of the tomogram (1) and normalized distribution function $P(\eta)$ of the set of parameters η , the joint probability distribution (11) will be automatically normalized, but in the space of two sets of variables x and η

$$\int \tilde{\mathcal{F}}(x, \eta, t) dx d\eta = 1. \quad (12)$$

Further let us make use of the universal star product scheme (see, e.g. [20]). For this we introduce the corresponding dequantizer and quantizer operators relating the function $\mathcal{F}(x, \eta, t)$ and the density matrix $\hat{\rho}(t)$. It is evident that

$$\tilde{\mathcal{F}}(x, \eta, t) = \text{Tr} \left\{ \hat{\rho}(t) \hat{U}_{\mathcal{F}}(x, \eta) P(\eta) \right\} = \text{Tr} \left\{ \hat{\rho}(t) \hat{U}_{\tilde{\mathcal{F}}}(x, \eta) \right\}, \quad (13)$$

$$\hat{\rho}(t) = \int \hat{D}_{\mathcal{F}}(x, \eta) P^{-1}(\eta) \tilde{\mathcal{F}}(x, \eta, t) dx d\eta = \int \hat{D}_{\tilde{\mathcal{F}}}(x, \eta) \tilde{\mathcal{F}}(x, \eta, t) dx d\eta, \quad (14)$$

where $\hat{U}_{\tilde{\mathcal{F}}}(x, \eta)$, $\hat{D}_{\tilde{\mathcal{F}}}(x, \eta)$ are the new dequantizer and quantizer for the star product scheme in the joint probability representation

$$\hat{U}_{\tilde{\mathcal{F}}}(x, \eta) = P(\eta) \hat{U}_{\mathcal{F}}(x, \eta), \quad \hat{D}_{\tilde{\mathcal{F}}}(x, \eta) = P^{-1}(\eta) \hat{D}_{\mathcal{F}}(x, \eta). \quad (15)$$

Additionally, in the definition of the symplectic joint probability distribution we will assume that the distribution function $P(\boldsymbol{\mu}, \boldsymbol{\nu})$ of tomographic parameters tends to zero at infinity with all of its derivatives, and it is integrable across the hyperspace $(\boldsymbol{\mu}, \boldsymbol{\nu})$ with any finite products of its arguments.

For the optical joint probability representation we note that the tomogram $w(\mathbf{x}, \boldsymbol{\theta}, t)$ contains the whole of the available information about the state when all of the components of the phase vector $\boldsymbol{\theta}$ are varied from zero to π , and it does not contain redundant information.

Therefore, in order that the joint distribution function $\tilde{w}(\mathbf{x}, \boldsymbol{\theta})$ should also contain all the information available on the state, the distribution function $P(\boldsymbol{\theta})$ must not turn to zero on the multitude $\{\theta_\sigma \in [0, \pi]\}$. Besides, the function $P(\boldsymbol{\theta})$ should be chosen so as to satisfy the normalization condition $\int_0^\pi P(\boldsymbol{\theta}) d^N \theta = 1$.

3 Correspondence rules for the operators, evolution equation and stationary states equation

If an operator \hat{A} defined on the set of density matrices $\{\hat{\rho}\}$ acts on $\hat{\rho}$ as $\hat{A}\hat{\rho}$, then, according to the general scheme, the action of this operator on $\tilde{\mathcal{F}}(x, \eta)$ in the joint probability representation can be expressed in terms of the operators $\hat{U}_{\tilde{\mathcal{F}}}$ and $\hat{D}_{\tilde{\mathcal{F}}}$ as follows (for brevity we shall omit the argument t , assuming that the function $\tilde{\mathcal{F}}(x, \eta)$ may depend on time):

$$\begin{aligned} [\hat{A}]_{\tilde{\mathcal{F}}} \tilde{\mathcal{F}}(x, \eta) &= \text{Tr} \left\{ \hat{U}_{\tilde{\mathcal{F}}}(x, \eta) \hat{A} \int \hat{D}_{\tilde{\mathcal{F}}}(x', \eta') \tilde{\mathcal{F}}(x', \eta') dx' d\eta' \right\}, \\ &= \int \text{Tr} \left\{ \hat{U}_{\tilde{\mathcal{F}}}(x, \eta) \hat{A} \hat{D}_{\tilde{\mathcal{F}}}(x', \eta') \right\} \tilde{\mathcal{F}}(x', \eta') dx' d\eta', \end{aligned} \quad (16)$$

that is, in this representation the operator $[\hat{A}]_{\tilde{\mathcal{F}}}$ is, generally speaking, an integral operator with the kernel

$$\mathcal{K}(x, \eta, x', \eta') = \text{Tr} \left\{ \hat{U}_{\tilde{\mathcal{F}}}(x, \eta) \hat{A} \hat{D}_{\tilde{\mathcal{F}}}(x', \eta') \right\}. \quad (17)$$

With the help of formulae (16) - (17) and knowing the expression for any operator in the density matrix representation we can find its expression in the joint probability representation. However, because of the simple relation between dequantizers and quantizers for the tomographic representation and for the joint probability representation (15), the correspondence rules can be found directly from the appropriate rules in the relevant tomographic representation. That is, if $[\hat{A}]_{\mathcal{F}}$ is the expression for the operator \hat{A} in the tomographic representation, then in the joint probability representation we, obviously, have:

$$[\hat{A}]_{\tilde{\mathcal{F}}} = P(\eta) [\hat{A}]_{\mathcal{F}} P^{-1}(\eta). \quad (18)$$

Thus, for example, for the position operator

$$[\hat{\mathbf{q}}]_{\tilde{\mathcal{F}}} = P(\eta) [\hat{\mathbf{q}}]_{\mathcal{F}} P^{-1}(\eta). \quad (19)$$

For the sum and for the product of two operators \hat{A} and \hat{B} we can write:

$$[\hat{A} + \hat{B}]_{\tilde{\mathcal{F}}} = [\hat{A}]_{\tilde{\mathcal{F}}} + [\hat{B}]_{\tilde{\mathcal{F}}}, \quad [\hat{A}\hat{B}]_{\tilde{\mathcal{F}}} = [\hat{A}]_{\tilde{\mathcal{F}}} [\hat{B}]_{\tilde{\mathcal{F}}}.$$

From these properties it follows that for any analytic function $R(\hat{A}_1, \hat{A}_2, \dots, \hat{A}_k)$ on the set of operators $\hat{A}_1, \hat{A}_2, \dots, \hat{A}_k$ the equality is fulfilled

$$[R(\hat{A}_1, \hat{A}_2, \dots, \hat{A}_k)]_{\tilde{\mathcal{F}}} = R([\hat{A}_1]_{\tilde{\mathcal{F}}}, [\hat{A}_2]_{\tilde{\mathcal{F}}}, \dots, [\hat{A}_k]_{\tilde{\mathcal{F}}}). \quad (20)$$

Thus, in most cases it is sufficient to find the correspondence rules for position and momentum operators.

For any operator \hat{A} its symbol $\tilde{\mathcal{F}}_{\hat{A}}(x, \eta)$ and dual symbol $\tilde{\mathcal{F}}_{\hat{A}}^{(d)}(x, \eta)$ are also found in accordance with the general scheme using dequantizer and quantizer (15)

$$\tilde{\mathcal{F}}_{\hat{A}}(x, \eta) = \text{Tr} \left\{ \hat{A} \hat{U}_{\tilde{\mathcal{F}}}(x, \eta) \right\}, \quad (21)$$

$$\tilde{\mathcal{F}}_A^{(d)}(x, \eta) = \text{Tr} \left\{ \hat{A} \hat{D}_{\tilde{\mathcal{F}}}(x, \eta) \right\}. \quad (22)$$

The average value of the operator \hat{A} in the state described by the joint probability distribution $\tilde{\mathcal{F}}(x, \eta)$ is determined by the dual symbol as follows:

$$\langle \hat{A} \rangle = \int \tilde{\mathcal{F}}_A^{(d)}(x, \eta) \tilde{\mathcal{F}}(x, \eta) dx d\eta. \quad (23)$$

The evolution equation for the joint probability distribution is found from the von-Neumann equation

$$i\hbar \partial_t \hat{\rho} = [\hat{H}, \hat{\rho}] \quad (24)$$

according to the method [13]

$$\partial_t \tilde{\mathcal{F}}(x, \eta, t) = \frac{2}{\hbar} \int \text{Im} \left[\text{Tr} \left\{ \hat{H}(t) \hat{D}_{\tilde{\mathcal{F}}}(x', \eta') \hat{U}_{\tilde{\mathcal{F}}}(x, \eta) \right\} \right] \tilde{\mathcal{F}}(x', \eta', t) dx' d\eta', \quad (25)$$

or

$$\partial_t \tilde{\mathcal{F}}(x, \eta, t) = \frac{2}{\hbar} \text{Im} \hat{H} ([\hat{\mathbf{q}}]_{\tilde{\mathcal{F}}}, [\hat{\mathbf{p}}]_{\tilde{\mathcal{F}}}, t) \tilde{\mathcal{F}}(x, \eta, t). \quad (26)$$

where hereinafter ∂_t is an abbreviated designation of the derivative $\partial/\partial t$.

When the Hamiltonian is time-independent, for the stationary states equation

$$\hat{H} \hat{\rho}_E = E \hat{\rho}_E = \hat{\rho}_E \hat{H} \quad (27)$$

in the joint probability representation we have:

$$E \tilde{\mathcal{F}}_E(x, \eta) = \text{Re} \hat{H} ([\hat{\mathbf{q}}]_{\tilde{\mathcal{F}}}, [\hat{\mathbf{p}}]_{\tilde{\mathcal{F}}}) \tilde{\mathcal{F}}_E(x, \eta). \quad (28)$$

Joint probability distributions $\tilde{\mathcal{F}}_E(x, \eta)$ corresponding to the stationary states must also satisfy the stationary condition, which can be written as:

$$\text{Im} \hat{H} \{ [\hat{\mathbf{q}}]_{\tilde{\mathcal{F}}}, [\hat{\mathbf{p}}]_{\tilde{\mathcal{F}}} \} \tilde{\mathcal{F}}_E(x, \eta) = 0. \quad (29)$$

4 Symplectic joint probability representation with the Gaussian distribution of tomographic parameters $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$

As an example, consider the case with the distribution of the tomographic parameters $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ in the form of shifted and deformed Gaussian function for N -dimensional quantum system

$$P_1(\boldsymbol{\mu}, \boldsymbol{\nu}) = \pi^{-N} \prod_{\sigma=1}^N \xi_{\sigma}^{-1} \zeta_{\sigma}^{-1} \exp \left[-\frac{(\mu_{\sigma} - \mu_{0\sigma})^2}{\xi_{\sigma}^2} \right] \exp \left[-\frac{(\nu_{\sigma} - \nu_{0\sigma})^2}{\zeta_{\sigma}^2} \right]. \quad (30)$$

From the correspondence rules for the operators of components of position and momentum in the symplectic tomography representation [11]

$$[\hat{q}_j]_M = -\partial_{\mu_j} \partial_{X_j}^{-1} + i \frac{\nu_j \hbar}{2} \partial_{X_j}, \quad [\hat{p}_j]_M = -\partial_{\nu_j} \partial_{X_j}^{-1} - i \frac{\mu_j \hbar}{2} \partial_{X_j}, \quad (31)$$

with the help of formula (18) we obtain the operators of position and momentum in the joint probability representation

$$[\hat{q}_j]_{\widetilde{M}} = P_1(\boldsymbol{\mu}, \boldsymbol{\nu}) \left(-\partial_{X_j}^{-1} \partial_{\mu_j} + i \frac{\nu_j \hbar}{2} \partial_{X_j} \right) P_1^{-1}(\boldsymbol{\mu}, \boldsymbol{\nu}) = - \left(2 \frac{\mu_j - \mu_{0j}}{\xi_j^2} + \partial_{\mu_j} \right) \partial_{X_j}^{-1} + i \frac{\nu_j \hbar}{2} \partial_{X_j}, \quad (32)$$

$$[\hat{p}]_{\widetilde{M}} = - \left(2 \frac{\nu_j - \nu_{0j}}{\zeta_j^2} + \partial_{\nu_j} \right) \partial_{X_j}^{-1} + i \frac{\mu_j \hbar}{2} \partial_{X_j}. \quad (33)$$

where we introduced the designation for inverse derivatives [16]

$$\partial_{x_\sigma}^{-n} F(x_\sigma) = \frac{1}{(n-1)!} \int (x_\sigma - x'_\sigma)^{n-1} \Theta(x_\sigma - x'_\sigma) F(x'_\sigma) dx'_\sigma, \quad (34)$$

where $\Theta(x_\sigma - x'_\sigma)$ is a Heaviside step function.

For the creation \hat{a}_j and annihilation \hat{a}_j^\dagger operators in the symplectic tomography representation we know [11] that

$$\begin{aligned} [\hat{a}_j]_M &= \sqrt{\frac{m_j \omega_j}{2\hbar}} \left\{ \frac{\hbar}{2} \partial_{X_j} \left(\frac{\mu_j}{m_j \omega_j} + i\nu_j \right) - \partial_{X_j}^{-1} \left(\partial_{\mu_j} + \frac{i\partial_{\nu_j}}{m_j \omega_j} \right) \right\}, \\ [\hat{a}_j^\dagger]_M &= \sqrt{\frac{m_j \omega_j}{2\hbar}} \left\{ \frac{\hbar}{2} \partial_{X_j} \left(\frac{-\mu_j}{m_j \omega_j} + i\nu_j \right) - \partial_{X_j}^{-1} \left(\partial_{\mu_j} - \frac{i\partial_{\nu_j}}{m_j \omega_j} \right) \right\}. \end{aligned}$$

Consequently, in accordance with (18), in the joint probability representation we have:

$$[\hat{a}_j]_{\widetilde{M}} = \sqrt{\frac{m_j \omega_j}{2\hbar}} \left\{ \frac{\hbar}{2} \partial_{X_j} \left(\frac{\mu_j}{m_j \omega_j} + i\nu_j \right) - \partial_{X_j}^{-1} \left(\partial_{\mu_j} + i \frac{\partial_{\nu_j}}{m_j \omega_j} + 2 \frac{\mu_j - \mu_{0j}}{\xi_j^2} + i 2 \frac{\nu_j - \nu_{0j}}{m_j \omega_j \zeta_j^2} \right) \right\}, \quad (35)$$

$$[\hat{a}_j^\dagger]_{\widetilde{M}} = \sqrt{\frac{m_j \omega_j}{2\hbar}} \left\{ \frac{\hbar}{2} \partial_{X_j} \left(\frac{-\mu_j}{m_j \omega_j} + i\nu_j \right) - \partial_{X_j}^{-1} \left(\partial_{\mu_j} - i \frac{\partial_{\nu_j}}{m_j \omega_j} + 2 \frac{\mu_j - \mu_{0j}}{\xi_j^2} - i 2 \frac{\nu_j - \nu_{0j}}{m_j \omega_j \zeta_j^2} \right) \right\}. \quad (36)$$

It is easy to check that the equality $[[\hat{a}_j]_{\widetilde{M}}, [\hat{a}_j^\dagger]_{\widetilde{M}}] = 1$ is performed, because at the transition to the joint probability representation the commutation relations are maintained.

Using the general definition (22) one can obtain the dual symbols of any operators. Thus, for the identity operator $\hat{1}$, for the components \hat{q}_j and \hat{p}_j , and for the product $\hat{q}_j \hat{p}_j$ after some calculations we can write:

$$\widetilde{M}_1^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \pi^N \delta(\boldsymbol{\mu}) \delta(\boldsymbol{\nu}) \prod_{\sigma} \xi_{\sigma} \zeta_{\sigma} \exp \left(\frac{\mu_{0\sigma}^2}{\nu_{0\sigma}^2} + \frac{\nu_{0\sigma}^2}{\zeta_{\sigma}^2} \right), \quad (37)$$

$$\widetilde{M}_{\hat{q}_j}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = i \frac{\pi^N \sqrt{\hbar}}{\sqrt{m_j \omega_j}} \left[2 \frac{\mu_{0j}}{\xi_j^2} + \partial_{\mu_j} \right] \delta(\boldsymbol{\mu}) \delta(\boldsymbol{\nu}) \prod_{\sigma=1}^N \xi_{\sigma} \zeta_{\sigma} \exp \left(\frac{\mu_{0\sigma}^2}{\xi_{\sigma}^2} + \frac{\nu_{0\sigma}^2}{\zeta_{\sigma}^2} + i \sqrt{\frac{m_{\sigma} \omega_{\sigma}}{\hbar}} X_{\sigma} \right),$$

$$\widetilde{M}_{\hat{p}_j}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = i \frac{\pi^N \sqrt{\hbar}}{\sqrt{m_j \omega_j}} \left[2 \frac{\nu_{0j}}{\zeta_j^2} + \partial_{\nu_j} \right] \delta(\boldsymbol{\mu}) \delta(\boldsymbol{\nu}) \prod_{\sigma=1}^N \xi_{\sigma} \zeta_{\sigma} \exp \left(\frac{\mu_{0\sigma}^2}{\xi_{\sigma}^2} + \frac{\nu_{0\sigma}^2}{\zeta_{\sigma}^2} + i \sqrt{\frac{m_{\sigma} \omega_{\sigma}}{\hbar}} X_{\sigma} \right),$$

$$\begin{aligned} \widetilde{M}_{\hat{q}_j \hat{p}_j}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) &= \pi^N \left\{ \frac{-\hbar}{m_j \omega_j} \left[\frac{2\mu_{0j}}{\xi_j^2} \delta(\boldsymbol{\mu}) + \delta'_{\mu_j}(\boldsymbol{\mu}) \right] \left[\frac{2\nu_{0j}}{\zeta_j^2} \delta(\boldsymbol{\nu}) + \delta'_{\nu_j}(\boldsymbol{\nu}) \right] + \frac{i\hbar}{2} \delta(\boldsymbol{\mu}) \delta(\boldsymbol{\nu}) \right\} \\ &\times \prod_{\sigma=1}^N \xi_{\sigma} \zeta_{\sigma} \exp \left(\frac{\mu_{0\sigma}^2}{\xi_{\sigma}^2} + \frac{\nu_{0\sigma}^2}{\zeta_{\sigma}^2} + i \sqrt{\frac{m_{\sigma} \omega_{\sigma}}{\hbar}} X_{\sigma} \right). \end{aligned} \quad (38)$$

The dual symbol $\widetilde{M}_{\hat{A}}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$ of some operator \hat{A} according to formula (23) defines a linear continuous functional on the multitude of joint distribution functions $\widetilde{M}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$, that is, the set of $\widetilde{M}_{\hat{A}}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$ actually specifies a set of generalized functions on the multitude $\{\widetilde{M}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})\}$. It is clear that the equality of two different symbols of one operator must be treated as the equality of the generalized functions,

i.e., two symbols are equal each other if and only if for any joint distribution function we have the equality of the values of the functionals defined by these symbols. In that way, generally speaking, for any operator \hat{A} a set of symbols exist, which are equal in the sense of generalized functions (23).

Therefore, the dual symbols listed above are not uniquely defined, and we can write the other symbols of the same operators, for example, the dual symbol of the component of position operator from the physical meaning of the symplectic tomogram up to the normalization factor equals

$$\widetilde{M}_{\hat{q}_j}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) \sim X_j \delta(\mu_j - \xi_j) \delta(\boldsymbol{\nu}) \prod_{\sigma \neq j} \delta(\mu_\sigma). \quad (39)$$

The normalization factor is found from the equality

$$\langle \hat{q}_j \rangle = \int q_j W(\mathbf{q}, \mathbf{p}, t) d^N q d^N p = \int \widetilde{M}_{\hat{q}_j}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) P_1(\boldsymbol{\mu}, \boldsymbol{\nu}) M(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) d^N X d^N \boldsymbol{\mu} d^N \boldsymbol{\nu}.$$

After calculations we obtain the final result

$$\widetilde{M}_{\hat{q}_j}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \pi^N \zeta_j \exp \left(\frac{(\xi_j - \mu_{0j})^2}{\xi_j^2} + \frac{\nu_{0j}^2}{\zeta_j^2} \right) X_j \delta(\mu_j - \xi_j) \delta(\boldsymbol{\nu}) \prod_{\sigma \neq j} \xi_\sigma \zeta_\sigma \exp \left(\frac{\mu_{0\sigma}^2}{\xi_\sigma^2} + \frac{\nu_{0\sigma}^2}{\zeta_\sigma^2} \right) \delta(\mu_\sigma). \quad (40)$$

Similarly we find dual symbols of the components of momentum \hat{p}_j , of the product of components $\hat{q}_j \hat{p}_j$, and of the powers of components \hat{q}_j^n , \hat{p}_j^n :

$$\widetilde{M}_{\hat{p}_j}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \pi^N \xi_j \exp \left(\frac{\mu_{0j}^2}{\xi_j^2} + \frac{(\zeta_j - \nu_{0j})^2}{\zeta_j^2} \right) X_j \delta(\boldsymbol{\mu}) \delta(\nu_j - \zeta_j) \prod_{\sigma \neq j} \xi_\sigma \zeta_\sigma \exp \left(\frac{\mu_{0\sigma}^2}{\xi_\sigma^2} + \frac{\nu_{0\sigma}^2}{\zeta_\sigma^2} \right) \delta(\nu_\sigma), \quad (41)$$

$$\begin{aligned} \widetilde{M}_{\hat{q}_j \hat{p}_j}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) &= \frac{\pi^N}{2} X_j^2 \prod_{\sigma \neq j} \xi_\sigma \zeta_\sigma \exp \left(\frac{\mu_{0\sigma}^2}{\xi_\sigma^2} + \frac{\nu_{0\sigma}^2}{\zeta_\sigma^2} \right) \delta(\mu_\sigma) \delta(\nu_\sigma) \\ &\times \left\{ \delta(\mu_j - \xi_j) \delta(\nu_j - \zeta_j) \exp \left(\frac{(\xi_j - \mu_{0j})^2}{\xi_j^2} + \frac{(\zeta_j - \nu_{0j})^2}{\zeta_j^2} \right) - \delta(\mu_j - \xi_j) \delta(\nu_j) \exp \left(\frac{(\xi_j - \mu_{0j})^2}{\xi_j^2} + \frac{\nu_{0j}^2}{\zeta_j^2} \right) \right. \\ &\left. - \delta(\mu_j) \delta(\nu_j - \zeta_j) \exp \left(\frac{\mu_{0j}^2}{\xi_j^2} + \frac{(\zeta_j - \nu_{0j})^2}{\zeta_j^2} \right) \right\} + \frac{i\pi^N}{2} \delta(\boldsymbol{\mu}) \delta(\boldsymbol{\nu}) \prod_{\sigma} \xi_\sigma \zeta_\sigma \exp \left(\frac{\mu_{0\sigma}^2}{\xi_\sigma^2} + \frac{\nu_{0\sigma}^2}{\zeta_\sigma^2} \right), \end{aligned} \quad (42)$$

$$\widetilde{M}_{\hat{q}_j^n}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \pi^N \frac{\zeta_j}{\xi_j^{n-1}} \exp \left(\frac{(\xi_j - \mu_{0j})^2}{\xi_j^2} + \frac{\nu_{0j}^2}{\zeta_j^2} \right) X_j^n \delta(\mu_j - \xi_j) \delta(\boldsymbol{\nu}) \prod_{\sigma \neq j} \xi_\sigma \zeta_\sigma \exp \left(\frac{\mu_{0\sigma}^2}{\xi_\sigma^2} + \frac{\nu_{0\sigma}^2}{\zeta_\sigma^2} \right) \delta(\mu_\sigma), \quad (43)$$

$$\widetilde{M}_{\hat{p}_j^n}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \pi^N \frac{\xi_j}{\zeta_j^{n-1}} \exp \left(\frac{\mu_{0j}^2}{\xi_j^2} + \frac{(\zeta_j - \nu_{0j})^2}{\zeta_j^2} \right) X_j^n \delta(\boldsymbol{\mu}) \delta(\nu_j - \zeta_j) \prod_{\sigma \neq j} \xi_\sigma \zeta_\sigma \exp \left(\frac{\mu_{0\sigma}^2}{\xi_\sigma^2} + \frac{\nu_{0\sigma}^2}{\zeta_\sigma^2} \right) \delta(\nu_\sigma), \quad (44)$$

The dual symbol of the number of photons operator $\widetilde{M}_{\hat{a}_j^\dagger \hat{a}_j}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$ is expressed from the found symbols as follows:

$$\widetilde{M}_{\hat{a}_j^\dagger \hat{a}_j}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \frac{1}{2} \left(\frac{m_j \omega_j}{\hbar} \widetilde{M}_{\hat{q}_j^2}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) + \frac{1}{\hbar m_j \omega_j} \widetilde{M}_{\hat{p}_j^2}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) - \widetilde{M}_1^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) \right).$$

Thus, we obtained appearances of the dual symbols in the form of singular generalized functions. But in the joint probability representation as in the optical tomography representation [16] the dual symbols

of the operators can be expressed in the form of regular generalized functions. To calculate such symbols we preliminary note that according to (7)

$$\int \left[\prod_{\sigma=1}^N (X_{\sigma})^{\alpha_{\sigma}} \right] M(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) d^N X = \int \left[\prod_{\sigma=1}^N (\mu_{\sigma} q_{\sigma} + \nu_{\sigma} p_{\sigma})^{\alpha_{\sigma}} \right] W(\mathbf{q}, \mathbf{p}) d^N q d^N p. \quad (45)$$

Next, we consider the integral of the following form

$$\mathcal{I} = \int \mu_1^{\alpha_1} \dots \mu_N^{\alpha_N} \nu_1^{\beta_1} \dots \nu_N^{\beta_N} \frac{\partial^{k_1 + \dots + k_N + l_1 + \dots + l_N} P(\boldsymbol{\mu}, \boldsymbol{\nu})}{\partial \mu_1^{k_1} \dots \partial \mu_N^{k_N} \partial \nu_1^{l_1} \dots \partial \nu_N^{l_N}} d^N \mu d^N \nu. \quad (46)$$

If at least one $k_j > \alpha_j$ or $l_j > \beta_j$, then this integral is equal to zero. If $\alpha_j = k_j$ and $\beta_j = l_j$ for all $j = 1, \dots, N$, then

$$\mathcal{I} = \prod_{\sigma=1}^N (-1)^{k_{\sigma} + l_{\sigma}} k_{\sigma}! l_{\sigma}!. \quad (47)$$

Taking into account (45) and the specified properties of the integral (46) we can write

$$\begin{aligned} \int \left[\prod_{\sigma=1}^N q_{\sigma}^{k_{\sigma}} p_{\sigma}^{l_{\sigma}} \right] W(\mathbf{q}, \mathbf{p}) d^N q d^N p = \\ \int \left[\prod_{\sigma=1}^N (-1)^{k_{\sigma} + l_{\sigma}} \frac{X_{\sigma}^{(k_{\sigma} + l_{\sigma})}}{(k_{\sigma} + l_{\sigma})!} \right] \frac{\partial^{k_1 + \dots + k_N + l_1 + \dots + l_N} P(\boldsymbol{\mu}, \boldsymbol{\nu})}{\partial \mu_1^{k_1} \dots \partial \mu_N^{k_N} \partial \nu_1^{l_1} \dots \partial \nu_N^{l_N}} \frac{\widetilde{M}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})}{P(\boldsymbol{\mu}, \boldsymbol{\nu})} d^N X d^N \mu d^N \nu. \end{aligned} \quad (48)$$

Recall that for any operator \hat{A} acting on the density matrix you can find the appropriate operator in the Wigner-Weyl representation $[\hat{A}]_W$ acting on the Wigner function. It is well known that (see, e.g., [2])

$$\begin{aligned} \hat{q}_j \hat{p} &\leftrightarrow [\hat{q}_j]_W W(\mathbf{q}, \mathbf{p}) = \left(q_j + \frac{i\hbar}{2} \partial_{p_j} \right) W(\mathbf{q}, \mathbf{p}), \\ \hat{p}_j \hat{p} &\leftrightarrow [\hat{p}_j]_W W(\mathbf{q}, \mathbf{p}) = \left(p_j - \frac{i\hbar}{2} \partial_{q_j} \right) W(\mathbf{q}, \mathbf{p}). \end{aligned}$$

Therefore, the average value of any combination of the components of operators $\hat{\mathbf{q}}$ and $\hat{\mathbf{p}}$ can be expressed through a combination of integrals (48), and thus, we can find a regular symbol of any operator interesting to us. For example, let's find the regular symbol of the component \hat{q}_j ,

$$\begin{aligned} \langle \hat{q}_j \rangle &= \int \widetilde{M}_{\hat{q}_j}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) \widetilde{M}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) d^N X d^N \mu d^N \nu = \int q_j W(\mathbf{q}, \mathbf{p}) d^N q d^N p \\ &= - \int X_j \frac{\partial P(\boldsymbol{\mu}, \boldsymbol{\nu})}{\partial \mu_j} \frac{\widetilde{M}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})}{P(\boldsymbol{\mu}, \boldsymbol{\nu})} d^N X d^N \mu d^N \nu. \end{aligned} \quad (49)$$

Thus, we have

$$\widetilde{M}_{\hat{q}_j}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = - \frac{X_j}{P(\boldsymbol{\mu}, \boldsymbol{\nu})} \frac{\partial P(\boldsymbol{\mu}, \boldsymbol{\nu})}{\partial \mu_j}. \quad (50)$$

Substituting (30) instead of arbitrary distribution $P(\boldsymbol{\mu}, \boldsymbol{\nu})$ we obtain

$$\widetilde{M}_{\hat{q}_j}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = 2 \frac{\mu_j - \mu_{0j}}{\xi_j^2} X_j. \quad (51)$$

Direct verification shows that

$$\langle \hat{q}_j \rangle = \int 2 \frac{\mu_j - \mu_{0j}}{\xi_j^2} X_j P_1(\boldsymbol{\mu}, \boldsymbol{\nu}) M(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) d^N X d^N \mu d^N \nu = \int q_j W(\mathbf{q}, \mathbf{p}) d^N q d^N p.$$

Taking into account the symmetry considerations between the operators $\hat{\mathbf{q}}, \hat{\mathbf{p}}$ in the definition of the symplectic tomogram and the symmetry of the function $P_1(\boldsymbol{\mu}, \boldsymbol{\nu})$ from the symbol for position (51) we can obtain the dual symbol for the component of momentum

$$\widetilde{M}_{\hat{p}_j}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = 2 \frac{\nu_j - \nu_{0j}}{\zeta_j^2} X_j. \quad (52)$$

Similarly dual symbols for other operators in the form of regular generalized functions are calculated:

$$\widetilde{M}_{\hat{q}_j^2}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \frac{X_j^2}{\xi_j^4} [2(\mu_j - \mu_{0j})^2 - \xi_j^2], \quad (53)$$

$$\widetilde{M}_{\hat{p}_j^2}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \frac{X_j^2}{\zeta_j^4} [2(\nu_j - \nu_{0j})^2 - \zeta_j^2], \quad (54)$$

$$\widetilde{M}_{\hat{q}_j \hat{p}_j}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = 2X_j^2 \frac{\mu_j - \mu_{0j}}{\xi_j^2} \frac{\nu_j - \nu_{0j}}{\zeta_j^2} + \frac{i\hbar}{2}, \quad (55)$$

$$\widetilde{M}_{\hat{a}_j^\dagger \hat{a}_j}^{(d)}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \frac{X_j^2 m_j \omega_j}{2\hbar} \left[\frac{2(\mu_j - \mu_{0j})^2 - \xi_j^2}{\xi_j^4} + \frac{2(\nu_j - \nu_{0j})^2 - \zeta_j^2}{m_j^2 \omega_j^2 \zeta_j^4} \right] - \frac{1}{2}. \quad (56)$$

Dual symbols in the form of regular generalized functions are also defined non-uniquely, but all of their versions for the same operators are equal each other in the sense of generalized functions. E.g., the immediate verification shows that the symbols

$$\begin{aligned} \widetilde{M}_{\hat{q}_j^2}^{(d)} &= \frac{X_j^2}{2\xi_j^2} \left(\frac{3\mu_j^2}{\xi_j^2} - \frac{\nu_j^2}{\zeta_j^2} \right) \exp \left(-\mu_{0j} \frac{2\mu_j - \mu_{0j}}{\xi_j^2} - \nu_{0j} \frac{2\nu_j - \nu_{0j}}{\zeta_j^2} \right), \\ \widetilde{M}_{\hat{p}_j^2}^{(d)} &= \frac{X_j^2}{2\zeta_j^2} \left(\frac{3\nu_j^2}{\zeta_j^2} - \frac{\mu_j^2}{\xi_j^2} \right) \exp \left(-\mu_{0j} \frac{2\mu_j - \mu_{0j}}{\xi_j^2} - \nu_{0j} \frac{2\nu_j - \nu_{0j}}{\zeta_j^2} \right), \end{aligned}$$

give rise to the correct average values $\langle q_j^2 \rangle$ and $\langle p_j^2 \rangle$ respectively.

Further, let's write the evolution equation of the joint probability distribution. For this we use the previously found correspondence rules (32) and (33) for the components of the operators $\hat{\mathbf{q}}$ and $\hat{\mathbf{p}}$. From formula (26) with the Hamiltonian of the form

$$\hat{H} = \sum_{\sigma=1}^N \frac{\hat{p}_\sigma^2}{2m_\sigma} + V(\mathbf{q}, t) \quad (57)$$

after some calculations we obtain:

$$\partial_t \widetilde{M}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}, t) = \left[\sum_{j=1}^N \frac{\mu_j}{m_j} \left(2 \frac{\nu_j - \nu_{0j}}{\zeta_j^2} + \partial_{\nu_j} \right) + \frac{2}{\hbar} \text{Im} V([\hat{\mathbf{q}}]_{\widetilde{M}}, t) \right] \widetilde{M}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}, t). \quad (58)$$

The equation for the joint distributions of stationary states, when the potential V is time-independent, is found using (28)

$$\begin{aligned} & \left[E - \text{Re} V([\hat{\mathbf{q}}]_{\widetilde{M}}) \right] \widetilde{M}_E(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \\ & = \sum_{j=1}^N \left[\frac{\partial_{X_j}^2}{m_j} \left(2 \frac{(\nu_j + \nu_{0j})^2}{\zeta_j^4} + \frac{\partial_{\nu_j}^2}{2} + 2 \frac{\nu_j + \nu_{0j}}{\zeta_j^2} \partial_{\nu_j} + \frac{1}{\zeta_j^2} \right) - \frac{\mu_j^2 \hbar^2}{8m_j} \partial_{X_j}^2 \right] \widetilde{M}_E(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}). \end{aligned} \quad (59)$$

The distribution functions $\widetilde{M}_E(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$ also must satisfy by the stationary condition

$$\left[\sum_{j=1}^N \frac{\mu_j}{m_j} \left(\frac{\nu_j - \nu_{0j}}{\zeta_j^2} + \frac{\partial \nu_j}{2} \right) + \frac{1}{\hbar} \text{Im} V([\hat{\mathbf{q}}]_{\widetilde{M}}) \right] \widetilde{M}_E(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = 0, \quad (60)$$

which is obtained from equation (58) at $\partial_t \widetilde{M}_E(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = 0$.

5 Optical joint probability representation with the sum of Gaussian distributions for phase vector $\boldsymbol{\theta}$

Next, consider the representation of the joint distribution function $\widetilde{w}(\mathbf{X}, \boldsymbol{\theta})$ for N -dimensional quantum system, in which the distribution of the parameters $\boldsymbol{\theta}$ is assumed to be the following weighted sum:

$$P_2(\boldsymbol{\theta}) = \sum_{k=1}^K Q_k \mathcal{P}_k(\boldsymbol{\theta}), \quad \sum_{k=1}^K Q_k = 1. \quad (61)$$

where $\{Q_k\}$ is a set of weights, $\{\mathcal{P}_k(\boldsymbol{\theta})\}$ is a set of N -dimensional Gaussians

$$\mathcal{P}_k(\boldsymbol{\theta}) = \mathcal{N}_k \prod_{\sigma=1}^N \exp \left[-\frac{(\theta_\sigma - f_{k\sigma})^2}{\phi_{k\sigma}^2} \right]$$

with the normalization factors $\{\mathcal{N}_k\}$ such that

$$\int_0^\pi \mathcal{P}_k(\boldsymbol{\theta}) d^N \theta = \mathcal{N}_k \int_0^\pi \prod_{\sigma=1}^N \exp \left[-\frac{(\theta_\sigma - f_{k\sigma})^2}{\phi_{k\sigma}^2} \right] d^N \theta = 1.$$

As we noted above, $P(\boldsymbol{\theta}) \neq 0$ at $\{\theta_\sigma \in [0, \pi]\}$, and therefore the distribution $P_2(\boldsymbol{\theta})$ as a sum of Gaussians is the most universal distribution, because any other unequal to zero physical distribution can be represented in this form with prescribed precision.

From the correspondence rules for components of position and momentum operators in the optical tomography representation [12, 16]

$$[\hat{q}_j]_w = \sin \theta_j \partial_{X_j}^{-1} \partial_{\theta_j} + X_j \cos \theta_j + \frac{i\hbar}{2m_j \omega_j} \sin \theta_j \partial_{X_j}, \quad (62)$$

$$[\hat{p}_j]_w = m_j \omega_j \left(-\cos \theta_j \partial_{X_j}^{-1} \partial_{\theta_j} + X_j \sin \theta_j \right) - \frac{i\hbar}{2} \cos \theta_j \partial_{X_j}, \quad (63)$$

with the help of general formula (18) we can find the correspondence rules in the optical joint probability representation. To this end we note that each of expressions (62), (63) contains only one term non-commuting with $P(\boldsymbol{\theta})$. This term include the differentiation over the phase ∂_{θ_j} , and we should calculate the following expression:

$$P(\boldsymbol{\theta}) \partial_{\theta_j} P^{-1}(\boldsymbol{\theta}) = -P^{-1}(\boldsymbol{\theta}) [\partial_{\theta_j} P(\boldsymbol{\theta})] + \partial_{\theta_j}. \quad (64)$$

Substituting here $P(\boldsymbol{\theta}) = P_2(\boldsymbol{\theta})$, we obtain

$$P_2(\boldsymbol{\theta}) \partial_{\theta_j} P_2^{-1}(\boldsymbol{\theta}) = 2P_2^{-1}(\boldsymbol{\theta}) \sum_{k=1}^K Q_k \frac{\theta_j - f_{kj}}{\phi_{kj}^2} \mathcal{P}_k(\boldsymbol{\theta}) + \partial_{\theta_j}. \quad (65)$$

Thus, we can write

$$[\hat{q}_j]_{\widetilde{w}} = \sin \theta_j \partial_{X_j}^{-1} \left(2P_2^{-1}(\boldsymbol{\theta}) \sum_{k=1}^K Q_k \frac{\theta_j - f_{kj}}{\phi_{kj}^2} \mathcal{P}_k(\boldsymbol{\theta}) + \partial_{\theta_j} \right) + X_j \cos \theta_j + \frac{i\hbar}{2m_j \omega_j} \sin \theta_j \partial_{X_j}, \quad (66)$$

$$[\hat{p}_j]_{\tilde{w}} = m_j \omega_j \left[-\cos \theta_j \partial_{X_j}^{-1} \left(2P_2^{-1}(\boldsymbol{\theta}) \sum_{k=1}^K Q_k \frac{\theta_j - f_{kj}}{\phi_{kj}^2} \mathcal{P}_k(\boldsymbol{\theta}) + \partial_{\theta_j} \right) + X_j \sin \theta_j \right] - \frac{i\hbar}{2} \cos \theta_j \partial_{X_j}. \quad (67)$$

The dual symbols in the form of regular generalized functions in the joint probability representation are obtained from the relevant dual symbols for the optical tomogram found in [16] by the division on the distribution function $P(\boldsymbol{\theta})$. For example, for the average value of position

$$\langle \hat{q}_j \rangle = \int w_{\hat{q}_j}^{(d)}(\mathbf{X}, \boldsymbol{\theta}) w(\mathbf{X}, \boldsymbol{\theta}, t) d^N X d^N \theta = \int w_{\hat{q}_j}^{(d)}(\mathbf{X}, \boldsymbol{\theta}) P^{-1}(\boldsymbol{\theta}) \tilde{w}(\mathbf{X}, \boldsymbol{\theta}, t) d^N X d^N \theta.$$

Consequently $\tilde{w}_{\hat{q}_j}^{(d)}(\mathbf{X}, \boldsymbol{\theta}) = w_{\hat{q}_j}^{(d)}(\mathbf{X}, \boldsymbol{\theta}) P^{-1}(\boldsymbol{\theta})$, and so on:

$$\begin{aligned} \tilde{w}_{\hat{q}_j}^{(d)}(\mathbf{X}, \boldsymbol{\theta}) &= \frac{2}{\pi^N P(\boldsymbol{\theta})} X_j \cos \theta_j, & \tilde{w}_{\hat{p}_j}^{(d)}(\mathbf{X}, \boldsymbol{\theta}) &= \frac{2m_j \omega_j}{\pi^N P(\boldsymbol{\theta})} X_j \sin \theta_j, \\ \tilde{w}_{\hat{q}_j^2}^{(d)}(\mathbf{X}, \boldsymbol{\theta}) &= \frac{X_j^2}{\pi^N P(\boldsymbol{\theta})} (1 + 2 \cos 2\theta_j), & \tilde{w}_{\hat{p}_j^2}^{(d)}(\mathbf{X}, \boldsymbol{\theta}) &= \frac{X_j^2 m_j^2 \omega_j^2}{\pi^N P(\boldsymbol{\theta})} (1 - 2 \cos 2\theta_j), \\ \tilde{w}_{\hat{q}_j \hat{p}_j}^{(d)}(\mathbf{X}, \boldsymbol{\theta}) &= \frac{2m_j \omega_j}{\pi^N P(\boldsymbol{\theta})} X_j^2 \sin 2\theta_j + \frac{i\hbar}{2}. \end{aligned}$$

These formulae are correct for any distribution $P(\boldsymbol{\theta})$ unequal to zero including for $P_2(\boldsymbol{\theta})$.

The evolution equation for the joint probability distribution $\tilde{w}(\mathbf{X}, \boldsymbol{\theta}, t)$ is easily obtained from the evolution equation for the optical tomogram $w(\mathbf{X}, \boldsymbol{\theta}, t)$ found in Refs. [12], [13]

$$\partial_t w(\mathbf{X}, \boldsymbol{\theta}, t) = \left\{ \sum_{\sigma=1}^N \omega_{\sigma} \left[\cos^2 \theta_{\sigma} \partial_{\theta_{\sigma}} - \frac{1}{2} \sin 2\theta_{\sigma} (1 + X_{\sigma} \partial_{X_{\sigma}}) \right] + \frac{2}{\hbar} \text{Im } V([\hat{\mathbf{q}}]_w, t) \right\} w(\mathbf{X}, \boldsymbol{\theta}, t), \quad (68)$$

where components of the operator $[\hat{\mathbf{q}}]_w$ in the arguments of the potential V are given by the expression (62). With the help of (65) we have

$$\begin{aligned} \partial_t \tilde{w}(\mathbf{X}, \boldsymbol{\theta}, t) &= \left\{ \sum_{\sigma=1}^N \omega_{\sigma} \left[\cos^2 \theta_{\sigma} \left(2P_3^{-1}(\boldsymbol{\theta}) \sum_{k=1}^K Q_k \frac{\theta_{\sigma} - f_{k\sigma}}{\phi_{k\sigma}^2} \mathcal{P}_k(\boldsymbol{\theta}) + \partial_{\theta_{\sigma}} \right) \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \sin 2\theta_{\sigma} (1 + X_{\sigma} \partial_{X_{\sigma}}) \right] + \frac{2}{\hbar} \text{Im } V([\hat{\mathbf{q}}]_{\tilde{w}}, t) \right\} \tilde{w}(\mathbf{X}, \boldsymbol{\theta}, t), \end{aligned} \quad (69)$$

where components of the operator $[\hat{\mathbf{q}}]_{\tilde{w}}$ are given by (66).

The equation of the stationary states in the joint probability representation is also easily derived from the corresponding equation in the optical tomography representation found in Refs. [12], [13]

$$\begin{aligned} E w_E(\mathbf{X}, \boldsymbol{\theta}) &= \left[\sum_{\sigma=1}^N m_{\sigma} \omega_{\sigma}^2 \left\{ \frac{\cos^2 \theta_{\sigma}}{2} \partial_{X_{\sigma}}^{-2} (\partial_{\theta_{\sigma}}^2 + 1) - \frac{X_{\sigma}}{2} \partial_{X_{\sigma}}^{-1} (\cos^2 \theta_{\sigma} + \sin 2\theta_{\sigma} \partial_{\theta_{\sigma}}) \right. \right. \\ &\quad \left. \left. + \frac{X_{\sigma}^2}{2} \sin^2 \theta_{\sigma} - \frac{\hbar^2}{8m_{\sigma}^2 \omega_{\sigma}^2} \cos^2 \theta_{\sigma} \partial_{X_{\sigma}}^2 \right\} + \text{Re } V([\hat{\mathbf{q}}]_w) \right] w_E(\mathbf{X}, \boldsymbol{\theta}). \end{aligned} \quad (70)$$

Contrary to (68), this equation has the items with the double differentiation $\partial_{\theta_{\sigma}}^2$ not commuting with $P(\boldsymbol{\theta})$. Therefore, we should find the following expression:

$$P(\boldsymbol{\theta}) \partial_{\theta_j}^2 P^{-1}(\boldsymbol{\theta}) = \partial_{\theta_{\sigma}}^2 - 2 \frac{\partial_{\theta_{\sigma}} P(\boldsymbol{\theta})}{P(\boldsymbol{\theta})} \partial_{\theta_{\sigma}} + 2 \left(\frac{\partial_{\theta_{\sigma}} P(\boldsymbol{\theta})}{P(\boldsymbol{\theta})} \right)^2 - \frac{\partial_{\theta_{\sigma}}^2 P(\boldsymbol{\theta})}{P(\boldsymbol{\theta})}. \quad (71)$$

Using (64) and (71) from (70) we can obtain

$$\begin{aligned}
E \tilde{w}_E(\mathbf{X}, \boldsymbol{\theta}) = & \left[\sum_{\sigma=1}^N m_{\sigma} \omega_{\sigma}^2 \left\{ \frac{\cos^2 \theta_{\sigma}}{2} \partial_{X_{\sigma}}^{-2} \left[\partial_{\theta_{\sigma}}^2 - 2 \frac{\partial_{\theta_{\sigma}} P(\boldsymbol{\theta})}{P(\boldsymbol{\theta})} \partial_{\theta_{\sigma}} + 2 \left(\frac{\partial_{\theta_{\sigma}} P(\boldsymbol{\theta})}{P(\boldsymbol{\theta})} \right)^2 - \frac{\partial_{\theta_{\sigma}}^2 P(\boldsymbol{\theta})}{P(\boldsymbol{\theta})} + 1 \right] \right. \right. \\
& \left. \left. - \frac{X_{\sigma}}{2} \partial_{X_{\sigma}}^{-1} \left[\cos^2 \theta_{\sigma} + \sin 2\theta_{\sigma} \left(\partial_{\theta_{\sigma}} - \frac{\partial_{\theta_{\sigma}} P(\boldsymbol{\theta})}{P(\boldsymbol{\theta})} \right) \right] + \frac{X_{\sigma}^2}{2} \sin^2 \theta_{\sigma} - \frac{\hbar^2}{8m_{\sigma}^2 \omega_{\sigma}^2} \cos^2 \theta_{\sigma} \partial_{X_{\sigma}}^2 \right\} \right. \\
& \left. + \text{Re } V([\hat{\mathbf{q}}]_{\tilde{w}}) \right] \tilde{w}_E(\mathbf{X}, \boldsymbol{\theta}).
\end{aligned} \tag{72}$$

If the distribution $P(\boldsymbol{\theta})$ has only one peak

$$P(\boldsymbol{\theta}) = \mathcal{N} \prod_{\sigma=1}^N \exp \left[-\frac{(\theta_{\sigma} - f_{\sigma})^2}{\phi_{\sigma}^2} \right],$$

then formula (72) is converted to

$$\begin{aligned}
E \tilde{w}_E(\mathbf{X}, \boldsymbol{\theta}) = & \left[\sum_{\sigma=1}^N m_{\sigma} \omega_{\sigma}^2 \left\{ \frac{\cos^2 \theta_{\sigma}}{2} \partial_{X_{\sigma}}^{-2} \left[\partial_{\theta_{\sigma}}^2 + 4 \frac{\theta_{\sigma} - f_{\sigma}}{\phi_{\sigma}^2} \partial_{\theta_{\sigma}} + 4 \frac{(\theta_{\sigma} - f_{\sigma})^2}{\phi_{\sigma}^4} + \frac{2}{\phi_{\sigma}^2} + 1 \right] \right. \right. \\
& \left. \left. - \frac{X_{\sigma}}{2} \partial_{X_{\sigma}}^{-1} \left[\cos^2 \theta_{\sigma} + \sin 2\theta_{\sigma} \left(\partial_{\theta_{\sigma}} + 2 \frac{\theta_{\sigma} - f_{\sigma}}{\phi_{\sigma}^2} \right) \right] + \frac{X_{\sigma}^2}{2} \sin^2 \theta_{\sigma} - \frac{\hbar^2}{8m_{\sigma}^2 \omega_{\sigma}^2} \cos^2 \theta_{\sigma} \partial_{X_{\sigma}}^2 \right\} \right. \\
& \left. + \text{Re } V([\hat{\mathbf{q}}]_{\tilde{w}}) \right] \tilde{w}_E(\mathbf{X}, \boldsymbol{\theta}).
\end{aligned} \tag{73}$$

6 Conclusion

To summarize, we point out the main results of this work. The tomographic formulation of quantum mechanics based, for example, on optical tomographic probability distribution of quantum state, uses the conditional probability distribution depending on the local oscillator phase parameter. We expanded this approach studying properties of the joint probability distribution [17, 18], where the parameter is considered as extra random variable. We illustrated that the conventional quantum mechanics can be constructed in terms of such joint probability distributions where the quantum states are described by the functions dependent on random arguments and time, contrary to the tomographic representations, in which the tomographic parameters are not random, and contrary to Wigner [3], Husimi [4], and Glauber-Sudarshan [5, 6] representations, where the corresponding functions describing the states are not probability distributions at all. We presented the general formalism for symbols of operators in symplectic and optical joint probability representations. Taking the Gaussian functions as the distributions of the tomographic parameters we found the correspondence rules for most interesting physical operators and derived the expressions of the dual symbols of operators in the form of singular and regular generalized functions. Also we obtained evolution equations and stationary states equations for symplectic and optical joint probability distributions.

References

- [1] S. Mancini, V. I. Man'ko and P. Tombesi, *Phys. Lett. A*, **213**, 1 (1996).

- [2] A. Ibort, V. I. Man'ko, G. Marmo, A. Simoni, and F. Ventriglia, *Phys. Scr.*, **79**, 065013 (2009).
- [3] E. Wigner, *Phys. Rev.*, **40**, 749, (1932).
- [4] K. Husimi, *Proc. Phys.-Math. Soc. Japan*, **22**, 264-314 (1940).
- [5] R. J. Glauber, *Phys. Rev. Lett.*, **10**, 84-86 (1963).
- [6] E. C. G. Sudarshan, *Phys. Rev. Lett.*, **10**, No. 7, 277-279 (1963).
- [7] J. Bertrand and P. Bertrand, *Found. Phys.*, **17**, 397 (1987).
- [8] K. Vogel and H. Risken, *Phys. Rev. A*, **40**, 2847 (1989).
- [9] S. Mancini, V. I. Man'ko and P. Tombesi, *Quantum Semiclass. Opt.*, **7**, 615 (1995).
- [10] S. Mancini, V. I. Man'ko, and P. Tombesi, *Found. Phys.*, **27**, 801 (1997).
- [11] O. V. Man'ko and V. I. Man'ko, *J. Russ. Laser Res.*, **18**, 407 (1997).
- [12] Ya. A. Korennoy, V. I. Man'ko, *J. Russ. Laser Res.*, **32**, 74 (2011).
- [13] Ya. A. Korennoy, V. I. Man'ko, *J. Russ. Laser Res.*, **32**, 338 (2011).
- [14] Ya. A. Korennoy, V. I. Man'ko, *J. Russ. Laser Res.*, **36**, 534 (2015).
- [15] Ya. A. Korennoy, V. I. Man'ko, *Int. J. Theor. Phys.*, **55**, 4885 (2016).
- [16] G.G. Amosov, Ya. A. Korennoy, V. I. Man'ko, *Phys. Rev. A*, **85**, 052119 (2012).
- [17] M. A. Man'ko, V. I. Man'ko, *AIP Conf. Proc.*, **1488**, 110 (2012).
- [18] M. A. Man'ko, *Phys. Scr.*, **T153**, 014045 (2013).
- [19] Mr. Bayes, Mr. Price, *Phil. Trans.*, **53**, 370 (1763).
- [20] F. Lizzi, P. Vitale, *SIGMA*, **10**, 086 (2014).
- [21] O.V. Man'ko, V.I. Man'ko, and G. Marmo, *J. Phys. A* **35**, 699 (2002).